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Dynamical Systems, Topology, and Conductivity in Normal Metals

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We present here a complete description of all asymptotic regimes of conductivity in the so-called "Geometric Strong Magnetic Field limit" in the 3D single crystal normal metals with topologically complicated Fermi surfaces. In particular, new observable integer-valued characteristics of conductivity of topological origin were introduced by the present authors a few years ago; they are based on the notion of Topological Resonance which plays a basic role in the total picture. Our investigation is based on the study of dynamical systems on Fermi surfaces for the semi-classical motion of electrons in a magnetic field.

KEY WORDS: Normal metals; strong magnetic fields; conductivity; topology; Fermi surfaces.

1. INTRODUCTION

We consider the implications of the so called "Geometric Strong Magnetic Field limit" for the conductivity in normal metals with topologically complicated Fermi surface in the presence of a homogeneous magnetic field. The corresponding limit can be defined by the relation $1 \ll \omega_B \tau$. Here ω_B is the cyclotron frequency for the electron in crystal and τ is the mean free motion time between scattering acts. This theory is based on the use of a kinetic equation for the semiclassical electrons in a crystal in an external field. Let us say that the corresponding conditions for the external fields are always satisfied for the experimentally available electric and magnetic fields in the case of normal metals. We can speak, for example, about the limit of very strong magnetic fields in the experimental sense where the

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semiclassical approximation still gives the main features of transport phenomena. It works until the magnetic flux through the elementary cell of the ion lattice is small in comparison with the quantum unit. Taking into account the value of the physical parameters in the real single crystal normal metal (like gold, for example) we have finally $1t \ll B \ll 10^3 t$ for temperatures $T \geqslant 1K$. We will not discuss here any questions of rigorous foundations of this approach (very standard in the physics literature dedicated to the transport phenomena). The detailed explanations of this method can be found in classical books (see, for example refs. 7–10). Large numbers of physicists use it. Let us give here also the refs. 35 and 36 where the mathematically rigorous approach to the semiclassical motion of electrons in an electromagnetic field and lattice can be found. Indeed, no rigorous theory of the kinetic equation was developed yet, so while these papers are very good they don't make our results about conductivity more rigorous.

We will consider the electron states in a crystal parameterized by the energy bands and the quasimomentum $\mathbf{p}$ defined modulo the reciprocal lattice vectors. From the topological point of view we can say that quasimomentum belongs to the three-dimensional torus $\mathbb{T}^3$ (Brillouen zone) rather than to the Euclidean space $\mathbb{R}^3$. The torus $\mathbb{T}^3$ arises as factorization of the space $\mathbb{R}^3$ with respect to the reciprocal lattice. Topologists say that the space $\mathbb{R}^3$ is a covering over the 3-torus $\mathbb{T}^3$. The periodic dispersion relation $\epsilon(\mathbf{p})$ of any energy band can be considered as a one-valued continuous function on the torus $\mathbb{T}^3$. The Fermi surface $S_F\colon \epsilon(\mathbf{p}) = \epsilon_F$ can also be considered as a smooth compact two-dimensional surface without boundary embedded in the three-dimensional torus $\mathbb{T}^3$. In this paper we will often compare these two pictures in the 3-torus $\mathbb{T}^3$ and in the total Euclidean 3-space of quasimomenta. We will use the equation $\dot{\mathbf{p}} = \mathbf{F}_{\text{ext}}$ both in the torus $\mathbb{T}^3$ and in the covering Euclidean 3-space $\mathbb{R}^3$ for the homogeneous force $\mathbf{F}_{\text{ext}}$. In particular, we will consider the properties of the trajectories of this system both in these two spaces which will be very convenient for our consideration. Following the standard approach, we consider a system:

$$\dot{\mathbf{p}} = \frac{e}{c} [\nabla\epsilon(\mathbf{p}) \times \mathbf{B}] + e\mathbf{E}$$

for the semiclassical electron in both homogeneous electric and magnetic field. The value of electric field E is generally very small in experiments measuring the conductivity. Therefore only the trajectories of the system

$$\dot{\mathbf{p}} = \frac{e}{c} [\nabla\epsilon(\mathbf{p}) \times \mathbf{B}] \tag{1}$$

should be investigated in this approach. The trajectories of (1) in the Euclidean 3-space are given on the energy level $\epsilon(\mathbf{p}) = \text{const}$ by the plane sections orthogonal to magnetic field. So we have the analytic integrability of the system (1) in the 3-space R^3. However, the global structure of the trajectories on the 3-torus can be highly nontrivial after identification the quasimomenta equivalent modulo the reciprocal lattice.

The dynamical system (1) conserves also the volume element d^3p in $\mathbb{T}^3$ and does not change at all the Fermi distribution. So, in the absence of the electric field $\mathbf{E}$ we will have the electron distribution unchanged (up to the quantum corrections). Nevertheless, the response of this system to small perturbations will be completely different from the case $\mathbf{B} = 0$ and depend strongly on the geometry of trajectories of the dynamical system (1).

This dependence was first discovered by the school of I. M. Lifshitz (I. M. Lifshitz, M. Ya. Azbel, M. I. Kaganov, V. G. Peschanskii[1–3, 5, 6, 8]) in 1950's. Thus, in the work[1] the crucial difference in conductivity was found for the contribution of the closed and open periodic electron trajectories in $\mathbf{p}$-space considered as the total Euclidean 3-space $\mathbb{R}^3$. Namely, it was shown that the first case corresponds to the total decreasing of conductivity in the plane orthogonal to $\mathbf{B}$ for $B \to \infty$ while the second case corresponds to the strong anisotropy of conductivity in the plane orthogonal to $\mathbf{B}$ in the same limit: conductivity vanishes just in one direction in this plane only depending on the mean direction of the open periodic trajectory. In the works[2, 3] the interesting examples of Fermi surfaces and electron trajectories were considered. However the work[3] contains some conceptual mistake: open trajectories were found for the generic family of magnetic fields with different mean directions. This result is wrong. It contradicts to the "Topological Resonance" which is a base of our main results.[23, 28] We will discuss it in the Chapter 2.

The problem of classification of all possible trajectories on the Fermi surfaces was first set by S. P. Novikov[11] and then considered in his school (S. P. Novikov, A. V. Zorich, I. A. Dynnikov, S. P. Tsarev, A. Ya. Maltsev[12, 13, 16–34]).

The full classification of the conductivity tensors in the Geometric Strong Magnetic Field Limit (GSMF-limit) can be given now as a result of the topological studies of this important class of dynamical systems on the Fermi surfaces. The most important feature of this new picture is the invention of the observable "Topological numbers" in the conductivity which always appear in GSMF-limit in the situation when the conductivity in the plane orthogonal to the generic magnetic field $\mathbf{B}$ reveals the strong anisotropy for $B \to \infty$ which is stable with respect to the small rotations of the directions of $\mathbf{B}$. These Topological numbers have the form of the triples of integers $(m_1^\alpha, m_2^\alpha, m_3^\alpha)$. They describe some integral planes Γ^α in the reciprocal lattice. The directions of **B** for which the given triple $(m_1^\alpha, m_2^\alpha, m_3^\alpha)$ can be observed give always a region Ω_α of non-zero measure on the unit sphere. We call the region Ω_α on the unit sphere the "Stability zone" corresponding to the triple $(m_1^\alpha, m_2^\alpha, m_3^\alpha)$ which is constant within the domain Ω_α.

We claim that there are only two types of stable asymptotic behavior of conductivity in the GSMF-limit ($B \to \infty$) for any single crystal normal metal. Namely, the ("topologically nontrivial") case of the strongly anisotropic behavior of conductivity in the plane orthogonal to **B** corresponding to some triple of Topological numbers and the ("trivial") case of the uniform decreasing of conductivity in any direction orthogonal to **B** for $B \to \infty$. These cases cover the area on the 2-sphere of the full total measure, so the generic directions are either of the first type or of the second type. All other types of conductivity behavior in the GSMF-limit can not be stable with respect to small rotations of **B**. We don't give in this paper the proofs of these facts because of their rather high topological complexity; they can be found in the works.[11–13, 16–34] Let us introduce here notations for these stable situations which we will use in this paper.

Situation A (Topologically Trivial Behavior). It is the case of uniform decreasing of conductivity in the plane orthogonal to **B** for $B \to \infty$.

Situation B (Topological Numbers and Topological Resonance). This is the case of the strong anisotropy of conductivity in the plane orthogonal to **B** with decreasing in just one direction in this plane for $B \to \infty$. This direction can be described as the intersection of the plane orthogonal to magnetic field with some integral plane (given by two reciprocal lattice vectors). The corresponding integral plane remains unchanged under the small rotations of the magnetic field. Three integers characterizing this plane in the reciprocal lattice are exactly the observable topological numbers. Topological Resonance claiming that the mean directions of all open trajectories coincide for the generic magnetic field is a base of this result. It was extracted by the present authors from the core of the topological works quoted above. As it was already mentioned, the conceptual mistake has been made exactly here in the classical works of the Lifshitz group.

As we said above there are no other stable cases. However, for the complicated enough Fermi surfaces also highly nontrivial "chaotic" behavior of the conductivity tensor is possible (refs. 27, 28, 32, and 33) for the set of directions of the zero measure. The trajectories of this type were completely unknown in the classical literature. They were discovered recently in the topological works.(19, 25, 29) Chaotic trajectories can be divided into two different classes:

(1) Weakly chaotic trajectories (the Tsarev type);

(2) Strongly chaotic trajectories (the Dynnikov type).

The trajectories of the first kind can appear only if the direction of **B** is "partly rational," i.e., the plane $\Pi(\mathbf{B})$ orthogonal to **B** contains one (up to the multiplier) reciprocal lattice vector. The trajectories of the second kind can appear only if the direction of (**B**) is completely irrational, i.e., $\Pi(\mathbf{B})$ does not contain any reciprocal lattice vectors. In the case when the direction of **B** is purely rational (i.e., $\Pi(\mathbf{B})$ contains two linearly independent reciprocal lattice vectors) the chaotic electron trajectories can not appear.

The behavior of conductivity in GSMF-limit is very different for these two classes.(27, 28, 32, 33) Thus in the case of weakly chaotic trajectories the asymptotic expression of conductivity is just slightly different from the Situation B in the higher order terms; it corresponds to the strongly anisotropic behavior of conductivity in the plane $\Pi(\mathbf{B})$ for $B \to \infty$.(32, 33) This regime is unstable with respect to the small rotations of **B** unlike the regular case where the "Stability zones" can be observed.

The strongly chaotic trajectories, however, demonstrate completely different behavior of $\sigma^{ik}(B)$ in GSMF-limit.(27) Namely, in this case the conductivity in the plane orthogonal to **B** decreases as $B \to \infty$ (in all directions) with the different from the Situation A analytic dependence on B. Besides that, in this case the part of the Fermi surface is excluded from the conductivity along the direction of **B** for $B \to \infty$. The last fact leads to the "sharp minimum" in the conductivity along **B** for the given direction of **B** if the strongly chaotic trajectories appear. Usually the conductivity along **B** remains finite in this "sharp minima" since only a part of the Fermi surface plays role here. However, these minima can be observed (on the unit sphere)–see Chapter 3 for the more detailed information.

2. TOPOLOGICALLY STABLE CASES

To define the "Degree of irrationality" of magnetic field with respect to reciprocal lattice, *let* $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ *be the basis of the reciprocal lattice* Γ^* *such that the vectors of* Γ^* *are given by all possible integer linear combinations of* $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$. *Then*:

(1) *The direction of* **B** *is rational (or has irrationality* 1*) if the plane* $\Pi(\mathbf{B})$ *orthogonal to* **B** *contains two linearly independent reciprocal lattice vectors.*

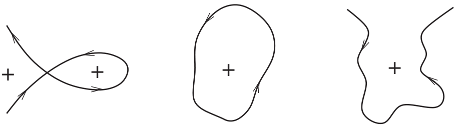

Fig. 1. The singular, compact and open non-singular trajectories. The signs "+" and "−" show the regions of larger and smaller values of $\epsilon(\mathbf{p})|_{\Pi}$ respectively.

(2) *The direction of* **B** *has irrationality* 2 *if the plane* $\Pi(\mathbf{B})$ *contains just one (up to multiplier) reciprocal lattice vector.*

(3) *The direction of* **B** *has irrationality* 3 *(or completely irrational) if the plane* $\Pi(\mathbf{B})$ *does not contain any reciprocal lattice vectors.*

The generic directions of the magnetic field are completely irrational. The direction of **B** should be "specially chosen" to have irrationality 1 or 2. We are going to consider now situations stable with respect to small rotations of **B**. This means in particular that specific features of such cases should not be connected with any kind of rationality of the direction of **B**, i.e., they should reveal all their properties for the completely irrational directions of magnetic field.

The electron trajectories are given by the intersections of the periodic Fermi surface with the family of parallel planes orthogonal to the magnetic field. For simplicity we will assume in this Chapter that the direction of **B** is completely irrational (for example, no open periodic trajectories can appear in the planes orthogonal to **B**). Let us postpone the specific (unstable) features of rational directions to the next Chapter.

We call the trajectory non-singular if it is not adjacent to the critical point. The trajectories adjacent to the critical points as well as the critical points themselves we call singular trajectories.

We call the non-singular trajectory compact if it is closed on the plane. We call the non-singular trajectory open if it is unbounded in $\mathbb{R}^2$.

The examples of singular, compact and open non-singular trajectories are shown on the Fig. 1a–c.

We call the open trajectory topologically regular (i.e., corresponding to the "topologically integrable" case) if it lies within the straight line strip of the finite width in $\mathbb{R}^2$ *and passes through it from* $-\infty$ *to* ∞ *(see Fig. 2a). All other open trajectories we call chaotic (Fig. 2b).*

Note that the topologically regular open trajectories are not periodic at all which would contradict to the irrationality of the direction of **B**. In fact they lie on some topological 2-tori (see below).[3]

[3] The ergodic properties of trajectories on the 2-tori were investigated in refs. 14 and 15.

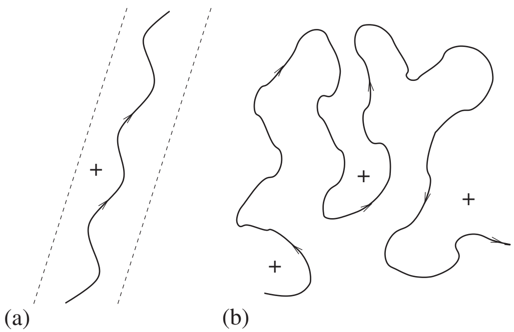

Fig. 2. "Topologically regular" (a) and "chaotic" (b) open trajectories in the plane Π orthogonal to **B**.

To introduce now the "Carriers of open trajectories" and the "Topological numbers", we follow the convenient description[25, 29] of the Fermi surface with the trajectories on it when the direction of **B** is fixed. We will be interested first of all in the open electron trajectories in the **p**-space. Let us say that in general just a part of the Fermi surface will be covered by the open electron trajectories. The remaining part will contain compact (or singular) trajectories. Let us remove all parts of the Fermi surface covered by the non-singular compact trajectories. The remaining part

$$S_F/(\text{Compact Nonsingular Trajectories}) = \bigcup_j S_j$$

is a union of the 2-manifolds S_j with boundaries ∂S_j who are the compact singular trajectories. The generic type in this case is a separatrix orbit with just one critical point like on the Fig. 3.

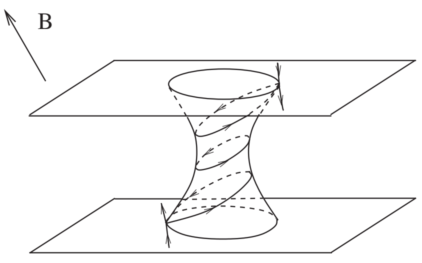

Fig. 3. The cylinder of compact trajectories bounded by the singular orbits. (The simplest case of just one critical point on the singular trajectory.)

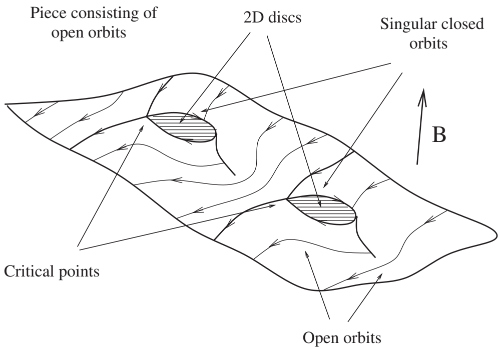

Fig. 4. The reconstructed constant energy surface with removed compact orbits and with the two-dimensional discs attached to the singular orbits; in the generic case there is just one critical point on every singular orbit.

We call every piece S_j the “**Carrier of open trajectories**.”

These pieces of Fermi surface, however, have holes with boundaries. They are not “closed manifolds” anymore. To get the closed manifolds let us make the next step:

We fill in the holes by the topological $2D$ discs in the planes orthogonal to $\mathbf{B}$; finally we are coming to the closed surfaces

$$\bar{S}_j = S_j \cup (2D \text{ discs})$$

(see Fig. 4).

This procedure gives the periodic surface $\bar{S}_F$ after the reconstruction and we can define the “compactified carriers of open trajectories” both in $\mathbb{R}^3$ and $\mathbb{T}^3$. Thus we have two representations of the reconstructed Fermi surface:

(1) The compact surface without boundary embedded in the space of quasimomenta $\mathbb{T}^3$ (consisting of several pieces without boundaries);

(2) The set of periodic two-dimensional surfaces without boundaries in the covering space $\mathbb{R}^3$.

Let us formulate now our main intermediate result which was established using the theorems extracted from the purely topological investigations (see, for example, ref. 12).

Fix the generic direction of $\mathbf{B}$ *and consider the set* $\{\bar{S}_j\}$ *carrying the open electron trajectories. Then the only two situations can be topologically stable with respect to the small rotations of* $\mathbf{B}$:

(A) *The set* $\{\bar{S}_j\}$ *is empty;*

(B) *The set* $\{\bar{S}_j\}$ *in the torus* $\mathbb{T}^3$ *consists of the even number of surfaces homeomorphic to the two-dimensional tori* $\mathbb{T}^2_j$*; all of them have the same homology class in* $H_2(\mathbb{T}^3)$ *up to the sign (sum of these classes is equal to zero). This property was called the "Topological resonance." The corresponding representation of the set* $\{\bar{S}_j\}$ *in total* **p***-space* $\mathbb{R}^3$ *can be described as follows:*

The manifolds $\bar{S}_j$ *represent the periodically deformed two-dimensional planes* $\Gamma_{(j)\alpha}$ *embedded in* $\mathbb{R}^3$ *with the same common integer-valued mean directions. In other words we have the set of periodically deformed (warped) integral planes in* $\mathbb{R}^3$ *which are all parallel in average and do not intersect each other. This picture remains unchanged after the small rotation of the magnetic field.*

Situation A corresponds to the absence of open electron trajectories on the Fermi level. We comment now on Situation B. We call the two-dimensional plane "integral" in $\mathbb{R}^3$ if it is generated by two reciprocal lattice vectors. This Topological resonance plays a leading role here as was first pointed out in refs. 23 and 28. The topological stability means in particular that corresponding picture remains the same after any rotation of **B** small enough: the number of connected components as well as the homological classes of corresponding tori $\mathbb{T}^2_j$ are the same for all directions of **B** close enough to the initial one. We make now the important physical conclusion from our main statement and consider the corresponding corollaries for the electrical conductivity.

It was also proved[29] that the total measure of the directions of **B** where different situations can arise is zero on the unit sphere for the generic Fermi surface S_F.

We will consider these two situations described above as the main basic foundation for the total classification of different regimes in the GSMF-limit for the generic case.

Define now the "Topological numbers" observable in situation B when we have regular open trajectories.

We call the "Topological numbers" corresponding to the stable open electron trajectories the triple of integers (m_1, m_2, m_3) *representing the integral 2-plane in the 3-space with reciprocal lattice. (Topologically it is a common homology class of the 2-tori* $\mathbb{T}^2_j$ in $\mathbb{T}^3$*.)*

This integers (m_1, m_2, m_3) can be extracted from common directions of periodically deformed two-dimensional planes representing $\{\bar{S}_j\}$ in $\mathbb{R}^3$ with respect to reciprocal lattice Γ^*. Namely, the planes Γ_α can be defined from the equation

$$m^1_\alpha[\mathbf{x}]_1 + m^2_\alpha[\mathbf{x}]_2 + m^3_\alpha[\mathbf{x}]_3 = 0$$

where $[\mathbf{x}]_i$ are the coordinates in the basis $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ of the reciprocal lattice, or equivalently

$$m^1_\alpha(\mathbf{x}, \mathbf{l}_1) + m^2_\alpha(\mathbf{x}, \mathbf{l}_2) + m^3_\alpha(\mathbf{x}, \mathbf{l}_3) = 0$$

where $\{\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3\}$ is the basis of the initial lattice in the coordinate space.

We can formulate now the main statement about the stable open trajectories in our approach:

All stable open electron trajectories have the topologically regular form, i.e., lie in the straight strips of the finite width in the planes orthogonal to **B** *in the* **p***-space and pass through them. All trajectories of this kind have the same mean directions for the given direction of* **B***: in average they are parallel to each other. The common direction of all these trajectories is given by the intersection of plane $\Pi(\mathbf{B})$ orthogonal to* **B** *with some integral plane Γ_α which is locally stable with respect to the small rotations of* **B**.

The fact that all topologically regular trajectories are parallel to each other expresses here the "Topological Resonance" property. It first appeared in refs. 23 and 28. It seems that nothing like that was known in the classical literature. In the work[3] for example the open electron trajectories with different mean directions were mistakably demonstrated for some analytic dispersion relations in the whole regions of the unit sphere parameterizing directions of **B**. We claim however, that this situation is completely impossible for any open region on the sphere. The important property of topologically regular open trajectories lies in the following fact: their contribution to the conductivity does not differ in the main order in the GSMF-limit from the (anisotropic) contribution of open periodic trajectories obtained in the old work.[1] It is very easy to prove this statement taking into account that the motion of electron is linear plus something bounded: one should simply repeat the essential arguments of this old work. The "Topological Resonance" claims more: all trajectories of this kind give the same anisotropy in the same coordinate system. Only this result makes this behavior experimentally observable. Let us present here corresponding expressions for the conductivity in the GSMF-limit for two situations described above.

Case A (Compact Trajectories Only).

$$\sigma^{ik} \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} & * \end{pmatrix}, \qquad \omega_B\tau \to \infty \qquad (2)$$

Case B (Open Topologically Regular Trajectories).

$$\sigma^{ik} \simeq \frac{ne^2\tau}{m^*}\begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & * & * \\ (\omega_B\tau)^{-1} & * & * \end{pmatrix}, \qquad \omega_B\tau \to \infty \tag{3}$$

Here $\simeq$ means "of the same order in $\omega_B\tau$ and * are some constants ~ 1. We assume here that the z-axis is always directed along the magnetic field **B** and the x-axis in the plane $\Pi(\mathbf{B})$ (orthogonal to **B**) is directed along the common mean direction of the topologically regular trajectories in **p**-space in the second case. Let us mention also that the relations (2)–(3) give only the order of magnitude of σ^{ik}.

The anisotropy of the tensor σ^{ik} in the formula (3) gives the experimental possibility of measuring the mean directions of the topologically regular open orbits for rather big values of B. Using the rotations of the direction of **B** it is possible also to find the "Stability zone" on the unit sphere and to determine the corresponding "Topological numbers" characterizing this stable situation. We see that there is only one direction $\hat{\eta}$ in the second case where the conductivity vanishes in the limit $B \to \infty$. According to the formula (3) this direction coincides with the common mean direction of the topologically regular trajectories in the **p**-space (i.e., orthogonal to the mean direction of these trajectories in the coordinate **x**-space). The direction $\hat{\eta}(\mathbf{B})$ depends on the direction of magnetic field. However, it varies in some integral plane Γ_α which is the same for the given "Stability zone." We can claim that the direction of conductivity decreasing $\hat{\eta} = (\eta_1, \eta_2, \eta_3)$ satisfies to the relation

$$m^1_\alpha(\hat{\eta}, \mathbf{l}_1) + m^2_\alpha(\hat{\eta}, \mathbf{l}_2) + m^3_\alpha(\hat{\eta}, \mathbf{l}_3) = 0$$

for all the points of stability zone Ω_α which makes possible the experimental observation of the numbers $(m^1_\alpha, m^2_\alpha, m^3_\alpha)$.

3. THE CHAOTIC CASES IN THE GSMF-LIMIT

Let us consider now the chaotic trajectories which can arise in the special cases for rather complicated Fermi surfaces. One should remember that they can appear only for a zero measure set of directions of the magnetic field. We think that for the generic Fermi surfaces the fractal (or Hausdorf) dimension of this set is strictly less than 1 (it was certainly proved by Dynnikov that it is no more than 1 for the generic Fermi surfaces, but it can be more for the nongeneric ones–see numerical studies in the works.[29, 31] Anyway, there is no proof of this until now.

We will first mention Tsarev's example of weakly chaotic trajectory having an asymptotic direction in $\mathbb{R}^3$.[19] We will not describe here the details of corresponding Fermi surface (see ref. 33). The trajectory of this kind can not be included in any straight strip of finite width in **p**-space. However this trajectory has always asymptotic direction in the plane orthogonal to **B**. The motion is linear plus smaller (but unbounded) terms. We can always choose the coordinate system such that the average values of the group velocities satisfy to the following condition:

$$\langle v_{\mathrm{gr}}^{x} \rangle = 0, \qquad \langle v_{\mathrm{gr}}^{y} \rangle \neq 0, \qquad \langle v_{\mathrm{gr}}^{z} \rangle \neq 0$$

Here again the z-axis coincides with the direction of **B** and the x-axis is directed along the asymptotic direction of the chaotic trajectory in **p**-space.

The behavior of conductivity in GSMF-limit does not coincide completely with the formula (3), however following formulae for the $\sigma^{ik}(B)$ can be proved:

$$\sigma^{ik}(\mathbf{B}) \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} o(1) & o(1) & o(1) \\ o(1) & * & * \\ o(1) & * & * \end{pmatrix}, \qquad \omega_B \tau \to \infty \tag{4}$$

which replaces the formula (3) for the case of weakly chaotic trajectories. Let us omit here all details o and just point out that the asymptotic direction of the weakly chaotic trajectory can be also observed experimentally. However, unlike the topologically regular case, the weakly chaotic trajectories are unstable with respect to generic small rotations of **B**. They correspond to some very small sets on the unit sphere. At last we say that the trajectories of this kind can appear only for the direction of **B** of irrationality 2, i.e., the plane $\Pi(\mathbf{B})$ should contain one reciprocal lattice vector in this situation.

The more interesting strongly chaotic trajectories do not have any asymptotic direction in $\mathbb{R}^3$ (see ref. 25). We just give the main features of such trajectories.

First of all, these trajectories can arise only in the case of magnetic field of irrationality 3. The carriers of such trajectories have the genus $\geqslant 3$. These trajectories are completely unstable with respect to the small rotations of **B**. They can be observed for the special fixed directions of **B** only in the case of the rather complicated Fermi surfaces. The approximate form of some trajectories of this kind is shown at Fig. 2b. Moreover, if the genus of Fermi surface is not very high ($g < 6$), the carrier of any strongly chaotic trajectory is invariant under the involution $\mathbf{p} \to -\mathbf{p}$ (after the appropriate choice of the initial point in $\mathbb{T}^3$). The ergodic theorem applied to the open trajectories on the carrier gives the relations:

$$\langle v_{\rm gr}^x \rangle = 0, \qquad \langle v_{\rm gr}^y \rangle = 0, \qquad \langle v_{\rm gr}^z \rangle = 0$$

for all three components of the group velocity on any of such trajectories. This important fact leads to the non-trivial behavior of corresponding contribution to the conductivity for $B \to \infty$. Namely we can show that all components of the corresponding contribution to $\sigma^{ik}(\mathbf{B})$ actually tends to zero in the limit $B \to \infty$.[27] We can write for this contribution:

$$\sigma^{ik}(\mathbf{B}) \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} o(1) & o(1) & o(1) \\ o(1) & o(1) & o(1) \\ o(1) & o(1) & o(1) \end{pmatrix} \tag{5}$$

for $B \to \infty$.[4]

We see that the strongly chaotic trajectories give the decreasing contribution for conductivity even along the magnetic field **B** (for rather big values of B). In the work[27] the special "scaling" asymptotic behavior of $\sigma^{ik}(\mathbf{B})$ were suggested. However, the full conductivity tensor includes also the contribution of compact (closed) electron trajectories having the form (2); it presents in all cases described above. So the strongly chaotic behavior does not remove completely the conductivity along the magnetic field **B** because of the contribution of compact trajectories. However, the sharp local minimum in this conductivity can be observed in this case.

It can be proved (see ref. 29) that for the generic Fermi surfaces the measure of directions of magnetic field **B** where the strongly chaotic behavior can be found on the Fermi surface is equal to zero. However, the total set on the unit sphere corresponding to the strongly chaotic trajectories of this kind can be rather complicated set with the non-trivial Hausdorf dimension. We expect that the Hausdorf dimension of this set is strictly less than 1 for the generic Fermi surfaces. For the nongeneric cases it might be even more than 1.

At last let us say that we expect that either the small stability zones or the strongly chaotic trajectories in fact were observed in the experimental data represented in ref. 4 (see refs. 27 and 28). However these data are not detailed enough (for example the conductivity along magnetic field was not measured in this experiments).

Let us describe now the total picture for the angle diagram of conductivity in normal metal in the case of geometric strong magnetic field

[4] Actually the component $\sigma^{zz}(\mathbf{B})$ contains a non-vanishing term of the order of T^2/ϵ_F^2 for $B \to \infty$ for non-zero temperatures.[27] However, this parameter is very small for the normal metals; we don't take it here into account.

limit.[32, 33] Namely, we can observe the following objects on the unit sphere parameterizing the directions of **B**:

(1) The "stability zones" Ω_α corresponding to topologically regular open trajectories and parameterized by some integral planes Γ_α in the reciprocal lattice ("Topological Numbers"). All "stability zones" have the piecewise smooth boundaries on S^2.

The corresponding behavior of conductivity is described by the formula (3) and reveals the strong anisotropy in the planes orthogonal to the magnetic field. For the complicated Fermi surfaces we can observe also the "sub-boundaries" of the stability zones where the coefficients in (3) can have the sharp "jump" but do not change the "Topological Numbers" characterizing the "Stability zone" Ω_α.

(2) The net of the one-dimensional curves containing directions of irrationality $\leqslant 2$ where the additional periodic open trajectories in **p**-space can appear. The corresponding parts of the net are always the parts of the big (passing through the center of S^2) circles orthogonal to some reciprocal lattice vector. The asymptotic behavior of conductivity is given again by the formula (3) but unstable with respect to the small rotations of **B** going out from the corresponding curves.

(3) The "Special rational directions." We call the special rational direction the direction of **B** orthogonal to the integral plane Γ_α corresponding to some stability zone Ω_α in case when this direction belongs to the same stability zone on the unit sphere. We don't discuss here all the specific features which can appear in this situation and just say that some specialties can arise here, see refs. 32 and 33 where all corresponding possibilities are discussed.

(4) The weakly chaotic open orbits (**B** of irrationality 2). We can have points on the unit sphere where the open orbits are weakly chaotic. All open trajectories still have the asymptotic direction in this case and the conductivity reveals the strong anisotropy in the plane orthogonal to **B** as $B \to \infty$. The B dependence, however is slightly different from the formula (3) in this case.

(5) The strongly chaotic open orbits (**B** of irrationality 3).

For some points on S^2 we can have the strongly chaotic open orbits on the Fermi surface. At these points the local minimum of conductivity along the magnetic field is expected. The conductivity along **B** however remains finite as $B \to \infty$ in general situation because of the contribution of compact trajectories.

(6) At last we can have the open regions on the unit sphere where only the compact trajectories on the Fermi level are present (Situation A).

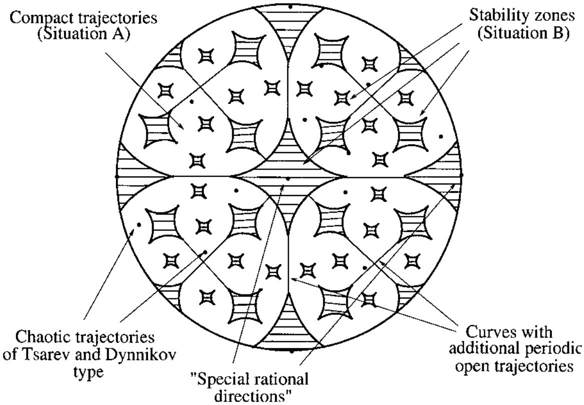

Fig. 5. The schematic representation of possible regimes for the different directions of the magnetic field **B** on the unit sphere.

The asymptotic behavior of conductivity tensor is given then by the formula (2).

In Fig. 5 we show the schematic picture of the regimes described above for different directions of the magnetic field **B**.

Some new features connected with the "magnetic breakdown" (self-intersecting Fermi surfaces) can be observed for rather strong magnetic fields. Up to this point it has been assumed throughout that different parts of the Fermi surface do not intersect each other. However, it is possible for some special lattices that the different components of the Fermi surface (parts corresponding to different conductivity bands) come very close to each other and may have an effective "reconstruction" as a result of the "magnetic breakdown" in the strong magnetic field limit. In this case we can have a situation in which the electron motion on the pieces of Fermi surface intersect each other. However, the intersections with other pieces do not affect at all the motion on one component. (The physical conditions for the corresponding values of B can be found in ref. 8). In this case the picture described above should be considered independently for the non-selfintersecting pieces of Fermi surface. We can have simultaneously several independent angle diagrams of this form on the unit sphere.

REFERENCES

1. I. M. Lifshitz, M. Ya. Azbel, and M. I. Kaganov, *Sov. Phys. JETP* **4**:41 (1957).
2. I. M. Lifshitz and V. G. Peschansky, *Soy. Phys. JETP* **8**:875 (1959).
3. I. M. Lifshitz and V. G. Peschansky, *Sov. Phys. JETP* **11**:137 (1960).
4. Yu. P. Gaidukov, *Sov. Phys. JETP* **10**:913 (1960).

5. I. M. Lifshitz and M. I. Kaganov, *Sov. Phys. Usp.* **2**:831 (1960).
6. I. M. Lifshitz and M. I. Kaganov, *Sov. Phys. Usp.* **5**:411 (1962).
7. C. Kittel, *Quantum Theory of Solids* (Wiley, New York, London, 1963).
8. I. M. Lifshitz, M. Ya. Azbel, and M. I. Kaganov, *Electron Theory of Metals* (Nauka, Moscow, 1971). [Translated: Consultants Bureau, New York, 1973.]
9. J. M. Ziman, *Principles of the Theory of Solids* (Cambridge University Press, Cambridge, 1972).
10. A. A. Abrikosov, *Fundamentals of the Theory of Metals* (Nauka, Moscow, 1987). [Translated: North-Holland, Amsterdam, 1998.]
11. S. P. Novikov, *Russian Math. Surveys* **37**:1 (1982).
12. A. V. Zorich, *Russian Math. Surveys* **39**:287 (1984).
13. S. P. Novikov, *Proc. Steklov Inst. Math.* **1** (1986).
14. V. I. Arnold, *Functional Analysis and its Appilcations* **25**:2 (1991).
15. Ya. G. Sinai and K. M. Khanin, *Functional Analysis and Its Applications* **26**:3 (1992).
16. I. A. Dynnikov, *Russian Math. Surveys* **57**:172 (1992).
17. I. A. Dynnikov, *Russian Math. Surveys* **58** (1993).
18. I. A. Dynnikov, A proof of Novikov's conjecture on semiclassical motion of electron, *Math. Notes* **53**:495 (1993).
19. S. P. Tsarev, Private communication (1992–93).
20. S. P. Novikov, Quasiperiodic structures in topology, in *Proc. Conference "Topological Methods in Mathematics,"* dedicated to the 60th birthday of J. Milnor, June 15-22, S. U. N. Y. Stony Brook, 1991 (Perish, Houston, TX, 1993), pp. 223–233.
21. A. V. Zorich, *Proc. "Geometric Study of Foliations" (Tokyo, November 1993)*, T. Mizutani, *et al.*, eds. (World Scientific, Singapore, 1994), pp. 479–498.
22. S. P. Novikov, *Proc. Conf. of Geometry, December 15–26, 1993* (Tel Aviv University, 1995).
23. S. P. Novikov and A. Ya. Maltsev, *Zh. ETP Lett.* **63**:855 (1996).
24. I. A. Dynnikov, Surfaces in 3-Torus: Geometry of plane sections, *Proc. of ECM2, BuDA*, (1996).
25. I. A. Dynnikov, Semiclassical motion of the electron. A proof of the Novikov conjecture in general position and counterexamples, in *Solitons, Geometry, and Topology: On the Crossroad*, V. M. Buchstaber and S. P. Novikov, eds., American Mathematical Society Translations, Series 2, Advances in the Mathematical Sciences, Vol. 179 (1997).
26. I. A. Dynnikov and A. Ya. Maltsev, *ZhETP* **85**:205 (1997).
27. A. Ya. Maltsev, *ZhETP* **85**:934 (1997).
28. S. P. Novikov and A. Ya. Maltsev, *Physics-Uspekhi* **41**:231 (1998).
29. I. A. Dynnikov, *Russian Math. Surveys* **54**:21 (1999).
30. S. P. Novikov, *Russian Math. Surveys* **54**:1031 (1999).
31. R. D. Leo, *SIAM Journal on Applied Dynamical Systems* **2**:517–545 (2003).
32. A. Ya. Maltsev and S. P. Novikov, ArXiv: math-ph/0301033, *Bulletin of Braz. Math. Society, New Series* **34**:171–210 (2003).
33. A. Ya. Maltsev and S. P. Novikov, ArXiv: cond-mat/0304471.
34. A. Ya. Maltsev, Arxiv: cond-mat/0302014.
35. G. Panati, H. Spohn, and S. Teufel, *Phys. Rev. Lett.* **88**:250405 (2002).
36. G. Panati, H. Spohn, and S. Teufel, ArXiv: math-ph/0212041.